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A NONSTATIONARY AXISYMMETRIC MOTION OF GAS

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Axisymmetric nonstationary and irrotational motions of gas can be described by the system of equations [1],

$$r\frac{\partial a}{\partial t}+Nr\frac{\partial a}{\partial r}+T\frac{\partial a}{\partial\theta}+(\gamma-1)\,a\left[\frac{\partial(Nr)}{\partial r}+\frac{\partial T}{\partial\theta}+N+T\operatorname{ctg}\theta\right]=0, \tag{1}$$
$$\frac{\partial N}{\partial t}+\frac{\partial}{\partial r}[(N^2+T^2)/2+a/(\gamma-1)]=0,$$
$$\frac{\partial N}{\partial\theta}-\frac{\partial}{\partial r}(Tr)=0,\ a=\frac{dp}{d\rho},\ p=A\rho^{\gamma},$$

where N and T are the radial and the tangential components, respectively, of the gas velocity; a is the square of sound velocity; r, θ are the spherical coordinates.

A class of solutions will be found for system (1) assuming that the velocity components N, T depend on the angle θ and the time t only. It follows from the third equation in (1) that

$$N=f(\theta,t),\ T=f'_\theta(\theta,t). \tag{2}$$

By inserting N and T as given by (2) into the second equation of (1), an expression is obtained for the square of sound velocity in terms of the function $f(\theta, t)$

$$a=-(\gamma-1)\left[rf'_t+\psi(\theta,t)\right], \tag{3}$$

where $\psi(\theta, t)$ is an arbitrary function. The use of (2) and (3) reduces the first equation in (1) to the following system:

$$f=t\varphi(\theta)+x(\theta),\ \psi+A(\theta)t^2/2+B(\theta)t+\mu(\theta)=0, \tag{4}$$
$$(\gamma-1)\,\psi\left(2f+f'_\theta\operatorname{ctg}\theta+f''_{\theta\theta}\right)+f'_\theta\psi'_\theta=0,$$

where

$$A(\theta)=(2\gamma-1)\,\varphi^2+(\gamma-1)\,\varphi''_{\theta\theta}\varphi+\varphi'^2+(\gamma-1)\,\varphi\varphi'_\theta\operatorname{ctg}\theta;\quad B(\theta)=(2\gamma-1)\,x\varphi+(\gamma-1)\,x''_{\theta\theta}\varphi+x'_\theta\varphi'_\theta+(\gamma-1)\varphi x'_\theta\operatorname{ctg}\theta;$$

$\mu(\theta)$ is an arbitrary function. Since θ and t are independent variables, therefore (4) implies an overdetermined system of equations for finding $\varphi(\theta)$, $x(\theta)$, $\mu(\theta)$

$$(\gamma-1)\left(2\varphi+\varphi'_\theta\operatorname{ctg}\theta+\varphi''_{\theta\theta}\right)A+\varphi'_\theta A'_\theta=0, \tag{5}$$
$$(\gamma-1)\left(2x+x'_\theta\operatorname{ctg}\theta+x''_{\theta\theta}\right)A+x'_\theta A'_\theta+2B\,(\gamma-1)\left(2\varphi+\varphi'_\theta\operatorname{ctg}\theta+\varphi''_{\theta\theta}\right)+2\varphi'_\theta B'_\theta=0,$$
$$(\gamma-1)\left(2\varphi+\varphi'_\theta\operatorname{ctg}\theta+\varphi''_{\theta\theta}\right)\mu+\varphi'_\theta\mu'_\theta+B\,(\gamma-1)\left(2x+x'_\theta\operatorname{ctg}\theta+\right.$$
$$\left.+x''_{\theta\theta}\right)+x'_\theta B'_\theta=0,$$
$$(\gamma-1)(2x+x'_\theta\operatorname{ctg}\theta+x''_{\theta\theta})\,\mu+x'_\theta\mu'_\theta=0.$$

The consistency of (5) is now analyzed. By assuming that $\varphi'_\theta\neq 0$, $x'_\theta\neq 0$, one can eliminate from (5) A'_θ, B'_θ, μ'_θ. As a result, one arrives at a relation between x and φ,

$$\left[Ax'_\theta/\varphi'_\theta+2\mu\varphi'_\theta/x'_\theta-2B\right]\left[\Phi/\varphi'_\theta-X/x'_\theta\right]=0, \tag{6}$$

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where $\Phi = (2\varphi + \varphi'_\theta \operatorname{ctg}\theta + \varphi''_{\theta\theta})(\gamma - 1)$; $X = (2x + x'_\theta \operatorname{ctg}\theta + x''_{\theta\theta})(\gamma - 1)$.

By setting the second factor equal to zero, a differential relation is obtained between x and φ,

$$x = \varphi\left[c_1 \int \varphi^{-2} R d\varphi + c_2\right], \text{where } R = \exp\left[2\int \varphi(\varphi'_\theta)^{-1} d\theta\right]. \tag{7}$$

Under condition (7), system (5) is as follows:

for $A \neq 0$ one has

$$A\Phi + \varphi'_\theta A'_\theta = 0,\ B = c_3 A,\ \mu = c_4 A; \tag{8}$$

for $A = 0$ one has

$$(2\gamma - 1)\varphi^2 + (\gamma - 1)\varphi''_{\theta\theta}\varphi + \varphi'^2_\theta + (\gamma - 1)\varphi\varphi'_\theta \operatorname{ctg}\theta = 0, \tag{9}$$
$$\Phi B + \varphi'_\theta B'_\theta = 0,\ \Phi\mu + \varphi'_\theta \mu'_\theta = 0.$$

The second equation of (8) imposes an additional differential relation on x and φ, which together with A and B, can be given by

$$A\left[c_3 - x'_\theta/\varphi'_\theta\right] - \varphi\left[x - \varphi x'_\theta/\varphi'_\theta\right] = 0. \tag{10}$$

By substituting the value of x from (7) into (10), one arrives at an equation for φ

$$A\left[\frac{c_3 - c_2}{c_1} - \int \varphi^{-2} R d\varphi - R/\varphi\right] - \varphi R = 0.$$

By differentiating the above with respect to θ, one obtains the first equation (8),

$$A\Phi + \varphi'_\theta A'_\theta = 0.$$

Thus, system (5) under condition (7) is consistent.

System (9) is now analyzed. The additional differential relation between x and φ in system (9) determines the second equation. The expression for B from (4) under the condition (7) and for $A = 0$ is

$$B = -c_1 \varphi R.$$

By substituting this value of B into the second equation in (9), one finds

$$R\left[(2\gamma - 1)\varphi^2 + (\gamma - 1)\varphi''_{\theta\theta}\varphi + \varphi'^2_\theta + (\gamma - 1)\varphi'_\theta \varphi \operatorname{ctg}\theta\right] = 0,$$

which implies that (9) is consistent. Then one has $\mu = c_3 \varphi R$ as well.

The case is now considered when the first factor in (6) vanishes, i.e.,

$$\mu = B x'_\theta/\varphi'_\theta - (x'^2_\theta A)/(2\varphi'^2_\theta). \tag{11}$$

Inserting (11) in the last two equations of (5) one arrives at a system for x and φ

$$2B\left[X + F'_\theta x'_\theta/F\right] - A\left[FX + 2F'_\theta x'_\theta - \Phi F x'_\theta/\varphi'_\theta\right] + 2B'_\theta x'_\theta = 0, \tag{12}$$
$$B\left[\Phi F + \varphi'_\theta F'_\theta + X\right] - A\varphi'_\theta F F'_\theta + 2B'_\theta x'_\theta = 0,$$

where $F = x'_\theta/\varphi'_\theta$. By subtracting the first equation from the second, one finds that

$$\left[A x'_\theta/\varphi'_\theta - B\right]\left[(x''_{\theta\theta}\varphi'_\theta - x'_\theta\varphi''_{\theta\theta})/\varphi'_\theta + X - \Phi x'_\theta/\varphi'_\theta\right] = 0. \tag{13}$$

By setting the first factor equal to zero, one finds

$$B = A x'_\theta/\varphi'_\theta. \tag{14}$$

If one substitutes B from (14) into the second equation of (12), one arrives at the expression

$$A\left[F^2\Phi - FX + 2F'_\theta x'_\theta\right] = 0. \tag{15}$$

The equation $A = 0$ together with (14), (11) implies that $B = 0$, $\mu = 0$; the latter complies with the particular case of the condition (7). The vanishing of the second factor together with (15) yields an equation for x

$$(\gamma - 3)(x''_{\theta\theta}\varphi'_\theta - \varphi''_{\theta\theta}x'_\theta) + 2(\gamma - 1)(x\varphi'_\theta - \varphi x'_\theta) = 0,$$

the second equation for φ being given by (14). A consistent solution of these two equations is given by $x = c\varphi$, which also agrees with a particular case of condition (7).

If one sets the second factor in (13) equal to zero, one obtains a differential relation between x and φ; employing the latter, the second equation in (12) is given in the form

$$(x'_\theta \varphi - x\varphi'_\theta)[(1-2\gamma)A + (3\gamma-2)^2\varphi^2/\gamma] = 0.$$

The vanishing of the first factor implies that $x = c\varphi$. This relation between x and φ has previously been analyzed. The vanishing of the second factor results in an equation for φ,

$$[2\gamma - 1 - (3\gamma-2)^2/(2\gamma^2-\gamma)]\varphi^2 + \varphi'^2_\theta + (\gamma-1)\varphi''_{\theta\theta}\varphi + (\gamma-1)\varphi\varphi'_\theta \operatorname{ctg}\theta = 0,$$

which together with the first equation (5) admits a consistent solution only given in the form $\varphi = 0$.

The case of $\varphi'_\theta = 0$, $x'_\theta = 0$ is now analyzed. For $\varphi'_\theta = 0$ Eq. (5) implies that $\varphi = 0$. By using (4) one obtains the stationary state of the flow,

$$N = x(\theta),\ T = x'_\theta(\theta),\ a^2 = (\gamma-1)\mu(\theta).$$

Under the conditions of a stationary state, system (1) implies the Bernoulli–Euler integral which imposes a differential relation between x and μ given by $(x^2 + x'^2_\theta)/2 + \mu = \text{const}$. If this equation is taken into account together with the last equation in (5), one can bring the system (1) to a single equation for the function $x(\theta)$,

$$x''_{\theta\theta}(a^2 - x'^2_\theta) + (2a^2 - x'^2_\theta)x + a^2 x'_\theta \operatorname{ctg}\theta = 0,$$
$$a^2 = -(\gamma-1)(x^2 + x'^2_\theta)/2 + c,$$

which describes the axisymmetric conical gas flow; it was analyzed in [2, 3]. The case of $x'_\theta = 0$ $(x \neq 0)$ results in $\mu = 0$, $B = 0$, $A = 0$. For $x = 0$ one has $B = 0$, $\mu = c_1 A$.

Thus, system (5), fully analyzed for consistency, is described in the case of a nonstationary state of gas flows by the following functions of the variables r, θ, t:

$$N(\theta, t) = \varphi\left[t + c_2 + c_1\int \varphi^{-2} R\, d\varphi\right],$$
$$T(\theta, t) = \varphi'_\theta\left[t + c_2 + c_1\left(\int \varphi^{-2} R\, d\varphi + \varphi^{-1}R\right)\right],$$
$$a(\theta, t) = -(\gamma-1)\left[\varphi r + (c_3 t^2/2 + c_4 t + c_5)/(R\varphi'_\theta \sin\theta)^{(\gamma-1)}\right],$$

where φ can be found from the equation

$$[(2\gamma-1)\varphi^2 + (\gamma-1)\varphi\varphi'_\theta \operatorname{ctg}\theta + \varphi'^2_\theta + (\gamma-1)\varphi\varphi''_{\theta\theta}]\Phi +$$
$$+ [(2\gamma-1)\varphi^2 + (\gamma-1)\varphi\varphi'_\theta \operatorname{ctg}\theta + \varphi'^2_\theta + (\gamma-1)\varphi\varphi''_{\theta\theta}]'_\theta \varphi'_\theta = 0. \quad (16)$$

Two particular solutions of Eq. (16) are available. The first one is $\varphi = c_6 \cos(\theta + c_7)$. The other one can be found from the equation

$$(2\gamma-1)\varphi^2 + (\gamma-1)\varphi\varphi'_\theta \operatorname{ctg}\theta + \varphi'^2_\theta + (\gamma-1)\varphi\varphi''_{\theta\theta} = 0. \quad (17)$$

Equation (17) is transformed to a Legendre equation with the aid of the substitution $\varphi = y^{(\gamma-1)/\gamma}$. For the latter the general solution was found in [4] which for φ can be written as

$$\varphi = \left[c_6 F\left(-\frac{\nu}{2}, \frac{1+\nu}{2}, \frac{1}{2}, \cos^2\theta\right) + c_7\cos\theta \cdot F\left(\frac{1-\nu}{2}, 1+\frac{\nu}{2}, \frac{3}{2}, \cos^2\theta\right)\right]^{\frac{\gamma-1}{\gamma}},$$

where $\nu = (3\gamma-2)/2(\gamma-1)$; F is the hypergeometric function.

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